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Quantization Loss for Convolutional Decoding in Rayleigh-Fading Channels

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Abstract—This letter presents a theoretical analysis (based on tight upper bounds on the error probability) of quantization loss with integer metrics used for convolutional decoding in the Rayleigh-fading channel. Optimum configurations with respect to the generalized cutoff rate criterion are established for 2-bit, 3-bit and 4-bit quantizers, and corresponding losses with both de facto industry-standard 1/2-rate and associated punctured 3/4-rate codes are evaluated. Assuming optimized thresholds, 4-bit metrics are shown to incur only a small quantization loss. However, results also indicate that the loss is sensitive to suboptimum threshold spacing.

Index Terms—Convolutional codes, integer metrics, quantization, Rayleigh-fading channels.

I. INTRODUCTION

PRACTICAL convolutional decoding hardware demands that input metrics are quantized to a limited set of integer values. Without careful configuration, this can lead to a significant quantization loss, or increase in the signal-to-noise ratio (SNR) required to achieve a particular decoded error probability [1]. For convolutional codes, the generalized cutoff rate (GCR) is a reasonable criterion for quantizer design; Binshtok and Shamai (Shitz) [2] have reported analysis of optimum quantizers for the Rayleigh-fading channel in terms of the GCR and have evaluated the difference in achievable rates between decoding with continuous and quantized metrics. This letter seeks to address the lack of published literature containing evaluation of expected quantization losses in terms of error probability for the fading channel case.

In Section II a model of an antipodal system in a flat Rayleigh-fading channel is defined. Based on the GCR criterion, optimum quantizer spacings are established in Section III. Section IV then plots upper union bounds on the decoded bit error probability; these allow the theoretical loss associated with the optimized quantizer configurations to be estimated and provide useful indications of the impact of suboptimum quantization.

II. SYSTEM MODEL

The system model is illustrated in Fig. 1. The input is a random binary vector, \mathbf{U} , from which the rate- R convo-

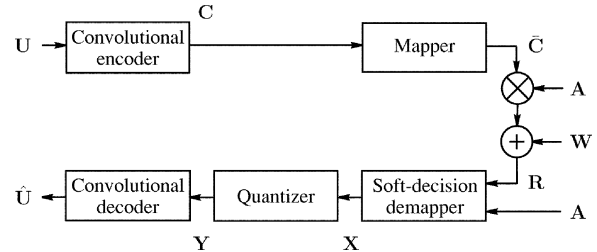


Fig. 1. System model.

lutional encoder produces the binary codeword, \mathbf{C} , with elements $C_i \in \{0, 1\}$. In this letter, we consider the defacto industry-standard 1/2-rate code with constraint length 7 and generator polynomials $(133, 171)_8$, and the 3/4-rate code formed by puncturing this code as in [3]. For each codeword element, the mapper produces a nonreturn-to-zero (NRZ) modulation symbol $\tilde{C}_i \in \{-1, 1\}$. The elements of the received vector, \mathbf{R} , are given by $R_i = \tilde{C}_i A_i + W_i$ where \mathbf{A} is a vector of i.i.d. Rayleigh distributed variables and \mathbf{W} is a vector of i.i.d. zero-mean Gaussian variables with variance σ^2 . It is assumed that $E\{A_i^2\} = 1$ and that $\sigma^2 = 1/2\langle\gamma\rangle$ where $\langle\gamma\rangle$ is the SNR.

With channel state information (CSI) available at the receiver, the soft-decision demapper produces optimum maximum-likelihood (ML) decoder inputs, \mathbf{X} , where $X_i = R_i A_i$. The quantizer converts this sequence to integer metric increments, \mathbf{Y} . Adopting the notation of [4], the output from a q -bit uniform quantizer, having $Q = 2^q$ regions with spacing ΔT is

$$Y_i = z, \quad \text{for } T_{z-1} < X_i \leq T_z \quad (1)$$

where the thresholds are

$$T_z = \begin{cases} -\infty, & \text{for } z = -1 \\ (z + 1 - 2^{q-1}) \Delta T, & \text{for } z = 0, 1, \dots, 2^q - 2 \\ \infty, & \text{for } z = 2^q - 1. \end{cases} \quad (2)$$

The ML decoder, which is typically implemented using the Viterbi algorithm, searches the code trellis to find the candidate codeword, or path, with maximum associated metric; this forms the estimate of the original data, $\hat{\mathbf{U}}$. Quantized metrics, L_i^Q , for each trellis branch are

$$L_i^Q(Y_i|C'_i) = \begin{cases} Q - 1 - Y_i, & \text{for } C'_i = 0 \\ Y_i, & \text{for } C'_i = 1 \end{cases} \quad (3)$$

where C'_i is the branch label, or candidate codeword element. These metrics are consistent with those outlined by Heller and Jacobs [5] and with many commercially available Viterbi decoder integrated circuits. Without quantization of the decoder

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inputs the continuous branch metrics, \mathbf{L}^∞ , have an alternative form

$$L_i^\infty(X_i|C'_i) = \bar{C}'_i X_i \quad (4)$$

where, as previously, \bar{C}'_i denotes a NRZ representation.

III. OPTIMUM QUANTIZER SPACING

We base quantizer design on the GCR of the discrete memoryless channel between the codeword, \mathbf{C} , and the quantized metric increments, \mathbf{Y} . Note that with respect to the GCR, uniformly spaced quantizers, as considered here, are optimum in the fading channel [2]. Following [6, p. 197], for binary modulation the GCR is

$$R_Q = 1 - \log_2 \{1 + D\}. \quad (5)$$

For the quantized metrics, \mathbf{L}^Q , we have

$$\begin{aligned} D &= \min_{\lambda \geq 0} E_{Y_i} \left\{ \exp \left(\lambda \left[L_i^Q(Y_i|0) - L_i^Q(Y_i|1) \right] \right) \middle| C_i = 1 \right\} \\ &= \min_{\lambda \geq 0} E_{Y_i} \left\{ \exp \left(\lambda [Q - 1 - 2Y_i] \right) \middle| C_i = 1 \right\} \\ &= \min_{\lambda \geq 0} e^{\lambda(Q-1)} \sum_{z=0}^{Q-1} e^{-2\lambda z} P_{Y|C}(z|1). \end{aligned} \quad (6)$$

The crossover probabilities $P_{Y|C}(z|1)$ in terms of the SNR, $\langle \gamma \rangle$, and the quantizer threshold spacing, ΔT , can then be determined from

$$P_{Y|C}(z|1) = f_{X|C}(T_z|1) - f_{X|C}(T_{z-1}|1) \quad (7)$$

where, for the system model considered here, the conditional cumulative density function of the decoder inputs, \mathbf{X} is given in [7] as

$$\begin{aligned} f_{X|C}(x|1) &= \begin{cases} \frac{\sigma^2}{\mu^2 + \mu} \exp \left\{ \frac{x\mu + x}{\sigma^2} \right\}, & \text{for } x \leq 0 \\ \frac{\sigma^2}{\mu^2 - \mu} \left(1 - \exp \left\{ \frac{-x\mu + x}{\sigma^2} \right\} \right) + \frac{\sigma^2}{\mu^2 + \mu}, & \text{for } x \geq 0 \end{cases} \end{aligned} \quad (8)$$

where $\mu = \sqrt{1 + 2\sigma^2}$. Using a numerical search based on these expressions, the threshold spacings which maximize R_Q have been obtained over a range of $\langle \gamma \rangle$ for 2-bit, 3-bit, and 4-bit quantizers. These are shown in Table I.

IV. UPPER BOUNDS ON ERROR PROBABILITY

The upper union bound on the bit error probability allows theoretical evaluation of the error performance of a coded communication system. For a rate $R = b/v$ convolutional code, this is

$$P_e \leq \frac{1}{b} \sum_{d=d_f}^{\infty} c_d P_d \quad (9)$$

where d_f and c_d are the free distance and the total number of information bits in error for all error events with starting point at distance d , respectively (which are well known for the codes considered here), and P_d is the pairwise error probability. It has been shown that this bound is tight for P_e smaller than around 10^{-4} with Rayleigh channel fading [8].

For continuous decoder metrics, \mathbf{L}^∞ , P_d is the probability of error for binary phase shift keying in the Rayleigh-fading channel with perfect CSI, maximal ratio combining and d th order diversity. This is given by Proakis [9, p. 723]. For quan-

TABLE I
QUANTIZER SPACING, ΔT , WHICH MAXIMIZES THE GENERALIZED CUTOFF RATE, R_Q , OVER A RANGE OF SNR, $\langle \gamma \rangle$; 2-BIT ($Q = 4$), 3-BIT ($Q = 8$) AND 4-BIT ($Q = 16$) METRICS

$\langle \gamma \rangle$ (dB)	$Q = 4$		$Q = 8$		$Q = 16$	
	ΔT	R_Q	ΔT	R_Q	ΔT	R_Q
0	0.645	0.34981	0.419	0.39298	0.257	0.40761
2	0.478	0.45291	0.306	0.50348	0.187	0.52001
4	0.348	0.55845	0.220	0.61318	0.133	0.63031
6	0.250	0.65733	0.156	0.71216	0.0933	0.72848
8	0.176	0.74289	0.109	0.79409	0.0644	0.80855
10	0.123	0.81216	0.0747	0.85723	0.0439	0.86927
12	0.0849	0.86541	0.0509	0.90326	0.0297	0.91283
14	0.0582	0.90482	0.0345	0.93550	0.0199	0.94284
16	0.0396	0.93325	0.0232	0.95745	0.0133	0.96294
18	0.0210	0.95204	0.0155	0.97212	0.00884	0.97615
20	0.0180	0.96753	0.0104	0.98180	0.00586	0.98472

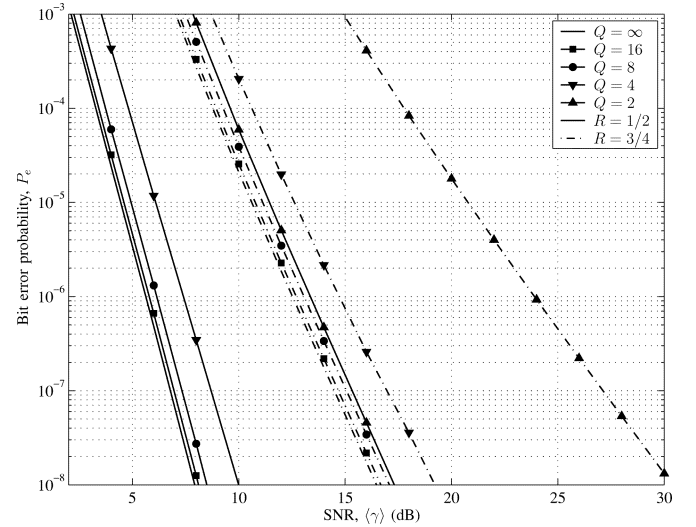


Fig. 2. Upper union bounds on bit error probability, P_e , versus SNR, $\langle \gamma \rangle$, with 1/2-rate and 3/4-rate codes based on the quantizer spacings, ΔT , in Table I; continuous ($Q = \infty$), 1-bit ($Q = 2$), 2-bit ($Q = 4$), 3-bit ($Q = 8$), and 4-bit ($Q = 16$) metrics.

tized metrics, \mathbf{L}^Q , Yasuda *et al.* provide a method for calculating P_d as a function of $P_{Y|C}(z|1)$ [10]. This crossover probability is defined by (7). As with R_Q in the previous section, the bound on P_e can, therefore, be evaluated analytically for any given ΔT and $\langle \gamma \rangle$. Yasuda's method involves determining, for every distance d in (9), all possible combinations of integers that satisfy a pair of nontrivial conditions; for greater than 4-bit quantization it is prohibitively difficult to compute.

Assuming the quantizer spacings given in Table I and using Yasuda's procedure for computing P_d , the bound on P_e in (9) has been evaluated over a range of $\langle \gamma \rangle$ with 2-bit, 3-bit and 4-bit metrics, and with both 1/2-rate and 3/4-rate codes. These results, along with the corresponding bounds assuming both 1-bit (for which $T_{-1} = -\infty$, $T_0 = 0$ and $T_1 = \infty$) and continuous metrics, are plotted in Fig. 2.

From this figure, assuming optimized quantizer thresholds, the losses relative to the continuous case for 2-bit, 3-bit, and 4-bit quantization are 1.78, 0.52, and 0.18 dB, respectively, with

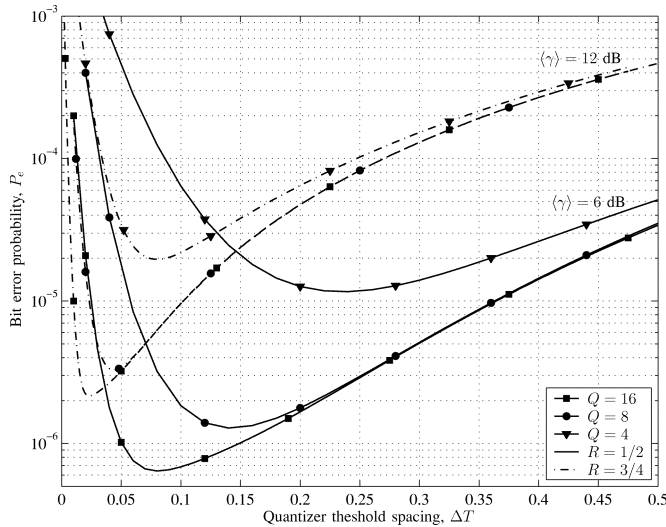


Fig. 3. Upper union bounds on bit error probability, P_e , versus quantizer spacing, ΔT , with 1/2-rate and 3/4-rate codes at SNR, $\langle \gamma \rangle = 6$ dB and $\langle \gamma \rangle = 12$ dB, respectively; 2-bit ($Q = 4$), 3-bit ($Q = 8$), and 4-bit ($Q = 16$) metrics.

the 1/2-rate code, and 2.19, 0.55, and 0.18 dB with the 3/4-rate code. These losses are measured at $P_e = 10^{-6}$, but remain approximately constant across the range of P_e plotted. For comparison, the use of 4-bit quantization in the Rayleigh-fading channel incurs a similar loss to 3-bit quantization in the AWGN channel (i.e., < 0.25 dB [1]). The results of Fig. 2 also indicate the importance of using soft-decision, as opposed to hard-decision, decoding in the fading channel; 1-bit quantization incurring a loss of 7.73 dB with the 1/2-rate code and 11.43 dB with the 3/4-rate code. The increased losses with the 3/4-rate code are reflective of the lower d_f , or reduced diversity gain, associated with the higher coding rate.

Fig. 3 plots the union bound of (9), again based on Yasuda's expression for P_d , as a function of quantizer spacing with 2-bit, 3-bit, and 4-bit metrics; results for 1/2-rate and 3/4-rate codes are plotted at SNRs of 6 and 12 dB, respectively. Note that although the optimized quantizer spacings in Table I (based on the GCR) do not correspond exactly to the values of ΔT that provide minimum P_e in Fig. 3 (based on the pairwise error probability), the disagreement is small. This highlights the value of the GCR criterion: exact error bounds such as those in Fig. 3 are difficult to compute whereas based on the GCR, near-optimal (with respect to P_e) quantizer designs can be straightforwardly developed in terms of the crossover probabilities $P_{Y|C}(z|1)$, which are dependent on the nature of the modulator, channel and demodulator rather than on the specific characteristics of the code employed.

The results of Fig. 3 also draw attention to the sensitivity of P_e (and hence the quantization loss) to suboptimum quantizer

design. This has important practical implications. Firstly, it implies that accurate Automatic Gain Control (AGC) is necessary in the receiver to ensure that the quantizer thresholds are appropriately positioned with respect to the continuous soft-decision metrics (here, quantizer design is based on the fact that $E\{A_i^2\} = 1$). Furthermore, it suggests that since, according to Table I, the optimum quantizer spacing varies significantly with SNR, fixed-level quantization is unlikely to yield good performance; instead the threshold spacing should be optimized according to the operating SNR of the receiver. In relation to this point, it is interesting to note from Table I the optimum ΔT varies less with $\langle \gamma \rangle$ as Q increases. This indicates that utilizing more quantization levels not only provides (if the optimum spacing can be accurately established) lower decoded error probability, but also yields (if nonoptimum fixed levels are employed) a quantizer that is more robust across a range of operating SNRs.

V. CONCLUSIONS

Through the analysis of upper bounds on the decoded bit error probability, 4-bit quantization with optimized thresholds has been shown to provide a loss of only 0.18 dB when integer metrics are used for convolutional decoding in the Rayleigh-fading channel. Results have also been presented which indicate that the error probability is sensitive to the use of suboptimum quantization.

REFERENCES

- [1] I. M. Onyszchuk, K.-M. Cheung, and O. Collins, "Quantization loss in Viterbi decoding," *IEEE Trans. Commun.*, vol. 41, pp. 261–265, Feb. 1993.
- [2] N. Binshtok and S. Shamai (Shitz), "Integer metrics for binary input symmetric output memoryless channels," *IEEE Trans. Commun.*, vol. 47, pp. 1636–1645, Nov. 1999.
- [3] D. Haccoun and G. Bégin, "High-rate punctured convolutional codes for Viterbi and sequential decoding," *IEEE Trans. Commun.*, vol. 37, pp. 1113–1125, Nov. 1989.
- [4] F. I. Alajaji and N. C. Phamdo, "Soft-decision COVQ for Rayleigh fading channels," *IEEE Commun. Lett.*, vol. 2, pp. 162–164, June 1998.
- [5] J. A. Heller and I. M. Jacobs, "Viterbi decoding for satellite and space communication," *IEEE Trans. Commun. Technol.*, vol. COM-19, pp. 835–848, Oct. 1971.
- [6] M. K. Simon, J. K. Omura, R. A. Scholtz, and B. K. Levitt, *Spread Spectrum Communications Handbook*: McGraw-Hill, 2002.
- [7] M. R. G. Butler, P. N. Fletcher, A. R. Nix, and D. R. Bull, "Quantization of Viterbi decoder soft-decision inputs for IEEE 802.11a in Rayleigh fading channels," in *4th Australian Communications Theory Workshop*, Melbourne, Australia, Feb. 2003, pp. 113–118.
- [8] P. K. Frenger, P. Orten, T. Ottosson, and A. B. Svensson, "Rate-compatible convolutional codes for multirate DS-CDMA systems," *IEEE Trans. Commun.*, vol. 47, pp. 828–836, June 1999.
- [9] J. G. Proakis, *Digital Communications*, 3rd ed. New York: McGraw-Hill, 1995.
- [10] Y. Yasuda, K. Kashiki, and Y. Hirata, "High-rate punctured convolutional codes for soft decision Viterbi decoding," *IEEE Trans. Commun.*, vol. COM-32, pp. 315–319, Mar. 1984.